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Diffusion on regular random fractals

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Abstract. We study random walks on structures intermediate to statistical and deterministic fractals called regular random fractals, constructed introducing randomness in the distribution of lacunas of Sierpinski carpets. Random walks are simulated on finite stages of these fractals and the scaling properties of the mean square displacement $\langle R_N^2\rangle^{1/2}$ of N-step walks are analysed. The anomalous diffusion exponents ν_w obtained are very near the estimates for the carpets with the same dimension. This result motivates a discussion on the influence of some types of lattice irregularity (random structure, dead ends, lacunas) on ν_w, based on results on several fractals. We also propose to use these and other regular random fractals as models for real self-similar structures and to generalize results for statistical systems on fractals.

1. Introduction

In the study of statistical systems on fractals, they are generally divided in two classes: deterministic (or regular) fractals, which have a definite rule of construction (like Sierpinski gaskets and carpets), and statistical fractals, whose fractal properties are obtained as an average (like percolation clusters or diffusion-limited aggregates—DLA). The former are very useful to search general properties of physical systems on fractals, while the latter are also expected to model real self-similar structures.

The study of random walks on fractal substrates has been intense in the last few years due to their applicability to a great variety of physical problems [1, 2]. One of their most important properties is the anomalous diffusion, i.e. the delay of the diffusion when compared to a Euclidean lattice [2]. Then the dimension of the random walk D_w is greater than 2 in most fractals ($D_w = 2$ in Euclidean lattices); it is defined by

$$\langle R_N^2\rangle \sim N^{2/D_w} \tag{1}$$

where $\langle R_N^2\rangle$ is the mean-square displacement of N-step walks.

In many deterministic finitely ramified fractals, D_w was obtained exactly [2, 3], and in some infinitely ramified fractals accurate estimates were obtained [4]. However, these results are somewhat disconnected from the properties of real systems because there have not been systematic studies on the effect of randomness on physical properties of statistical systems on fractals. On the other hand, most of the work in this area deal with percolation clusters [5, 6], where D_w is known with good accuracy, but they cannot represent many real fractal systems [6, 7].

The purpose of this work is to fill a gap in this field by studying random walks on a third class of fractals called regular random fractals [8], which are intermediate between

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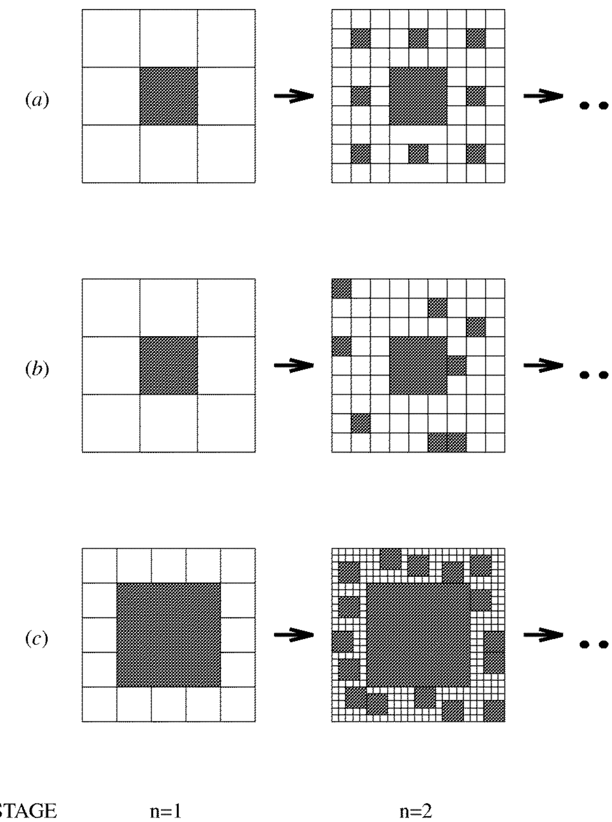

Figure 1. (a) Iterative construction of a Sierpinski carpet with $b = 3$ and $m = 1$; (b) construction of a regular random fractal with $b = 3$ and $m = 1$; stage $n = 1$ is the generator #1; (c) construction of a regular random fractal with $b = 5$, $m = 9$ (generator #2).

deterministic and statistical fractals. They are constructed by introducing randomness in the distribution of lacunas of a class of deterministic infinitely ramified fractals, the Sierpinski carpets. Figure 1(a) shows the iterative construction of a carpet with parameters $b = 3$ and $m = 1$. At each step of the construction, the squares are divided into b^2 subsquares and m of them are eliminated according to a fixed rule, defined by the generator (stage $n = 1$). The fractal obtained after an infinite number of iterations has fractal dimension

$$D_F = \frac{\ln(b^2 - m)}{\ln b}. \tag{2}$$

The corresponding regular random fractals are constructed eliminating the m subsquares randomly among the b^2 subsquares, as illustrated in figures 1(b) and (c). This rule defines an ensemble of fractals for each b and m, with fractal dimension also given by equation (1).

No previous study of physical systems on regular random fractals appears to be available, but there are some reasons to start these investigations. For example, the comparison with results on deterministic fractals will show the influence of randomness and the limitations of these fractals to represent real structures. Moreover, as they are infinitely ramified, it is interesting to compare these with results of the most intensively studied statistical fractals, percolation clusters and DLA, which are finitely ramified.

In section 2 of this paper we present the results of simulations of random walks on finite

Table 1. Fractal dimension D_F, critical exponent ν_w, the corresponding dimension D_w and correction to scaling constant A for random walks on regular random fractals and the exponent ν_w for the Sierpinski carpets with the same generators.

Generator	D_F	ν_w	D_w	A	ν_w (carpet)
1	1.8928	0.470 ± 0.005	2.13 ± 0.03	-1.0 ± 0.5	0.476 ± 0.005
2	1.7227	0.445 ± 0.005	2.25 ± 0.03	-1.5 ± 0.5	0.458 ± 0.004

stages of the construction of regular random fractals. It is shown that $\langle R_N^2 \rangle_L$, obtained in lattices of length L, obeys finite-size scaling and D_w is estimated. The comparison with results in the corresponding Sierpinski carpets will show that D_w is weakly affected by the random distribution of lacunas (section 3). We also compare the results for several fractals (statistical and deterministic) and present a discussion on the influence of geometric properties of D_w.

2. Numerical simulations

The generators of the ensembles of fractals studied here are shown in figures 1(*b*) and (*c*), numbered 1 and 2, respectively. Their fractal dimensions are shown in table 1. At each step of the construction of a fractal with generator 2 (figure 1(*c*)), the new lacunas with $m = 9$ subsquares are randomly distributed as blocks, like the lacuna with 1 subsquare of figure 1(*b*). In the regular fractals (figure 1(*a*)), the lattice sites were considered at the vertices of the non-eliminated squares, and non-active sites inside the lacunas. In the random lattices, the non-active sites are moved when the lacunas are distributed, but the sites at the border of the lacunas are still active.

We simulated random walks confined in stages 1–7 of fractals with generator 1 and stages 1–5 of fractals with generator 2 (the lattices have free edges). The characteristic length of stage n is $L = b^n$. The initial site of each walk is randomly chosen over the lattice and at each step the walker has equal probability to move to any neighbouring active site, up to N_{MAX} steps. Averaging over a certain number of initial sites (number of generated walks), we obtain $\langle R_N^2 \rangle_L$ for each N in a lattice of length L. This procedure is repeated for various members of each ensemble of fractals, each obtained from a different distribution of lacunas using the same generator. Averaging over these lattices we obtain the final estimates of $\langle R_N^2 \rangle_L$.

It was shown [4] that random walks on deterministic fractals (Sierpinski gaskets, carpets, pastry shells) obey the finite-size scaling hypothesis

$$\langle R_N^2 \rangle_L^{1/2} \approx L f(L N^{-\nu_w}) \tag{3}$$

where f is a generic function of its argument $x = L N^{-\nu_w}$, and

$$\nu_w = \frac{1}{D_w}. \tag{4}$$

Estimates of ν_w were obtained with accuracy around 1% from plots of $\langle R_N^2 \rangle_L^{1/2} / L$ versus x: the data for various lengths L collapse into a single curve ($f(x)$) when the correct value of ν_w is chosen. It parallels standard methods to calculate critical exponents of magnetic systems from results of simulations in finite lattices [9].

For the mean number of distinct visited sites $\langle S_N \rangle$ a similar finite-size scaling hypothesis holds, including a logarithmic correction in its asymptotic behaviour [10]. It was used to prove the Alexander–Orbach scaling relation $D_s = 2D_F/D_w$ [11] in the Sierpinski carpets and in the Eden trees (D_s is the spectral dimension), showing the importance of considering corrections to finite-size scaling relations in order to obtain accurate results.

For random walks on regular random fractals we propose an extension of relation (3) which includes a correction to scaling:

$$\langle R_N^2\rangle_L^{1/2} \approx L(1 + A/N)f(LN^{-\nu_w}) \tag{5}$$

where A is a constant. It is expected that $f(x) \to x^{-1}$ when $x \to \infty$, so that

$$\langle R_N^2\rangle \sim N^{\nu_w}(1 + A/N). \tag{6}$$

However, the analysis of the plots of $y = \langle R_N^2\rangle_L^{1/2}/L(1 + A/N)$ versus $x = LN^{-\nu_w}$ for several lengths L has to be more carefully done than in the deterministic lattices. The fractal properties of deterministic lattices are present in all lengthscales, and so those data collapse for x large as well as $x \approx 1$ (see [4] and [10]), which correspond to small N and very large N, respectively. But in random fractals we expect self-similarity to be observed only in lengths greater than the lattice parameter and less than the characteristic length of the whole structure. The first condition implies that the length of the walk ($\sim N^{\nu_w}$) must be greater than b, which is the smallest characteristic length of the structure. It is obtained by analysing only walks with $N > 50$ in all lattices. The second condition implies

$$N^{\nu_w} \ll L \qquad \Rightarrow \qquad x \gg 1 \tag{7}$$

where we only consider data with $x \gtrsim 10$.

These conditions limit the number of fractals we can analyse, because we need data from (at least) two stages n in the same range of x, and the simulations in large lattices become difficult. For example, the stage $n = 7$ of fractals with generator 1 are 2188×2188 square lattices with lacunas, where we simulated 4×10^5 walks with $N_{MAX} = 10^5$ steps to get reliable estimates of $\langle R_N^2\rangle_L^{1/2}$ (the accuracy is approximately 2% for the greatest lattices and better for small lattices).

In figures 2 and 3 we plot y against x for stages $n = 4$–7 of fractals with generator 1, using $\nu_w = 0.470$ and $A = -1$ (equation (5)). $\langle R_N^2\rangle_L^{1/2}$ was averaged over 40 different members of this ensemble, i.e. 40 different lattices for each length L.

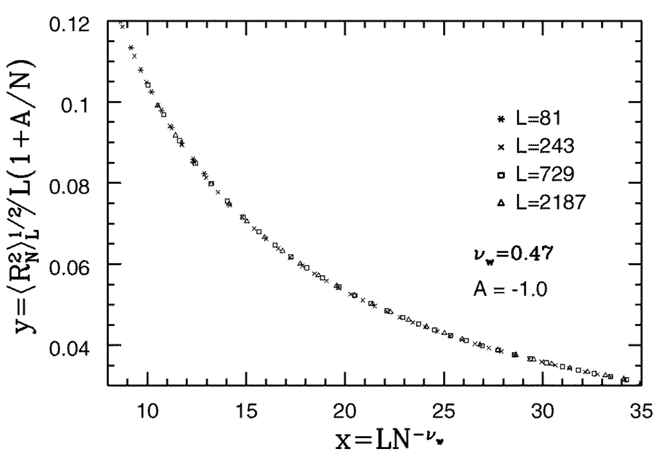

Figure 2. Plot of $y = \langle R_N^2\rangle_L^{1/2}/L(1 + A/N)$ against $x = LN^{-\nu_w}$ for random walks on finite stages of a fractal with generator 1, in the range $8 \leqslant x \leqslant 35$.

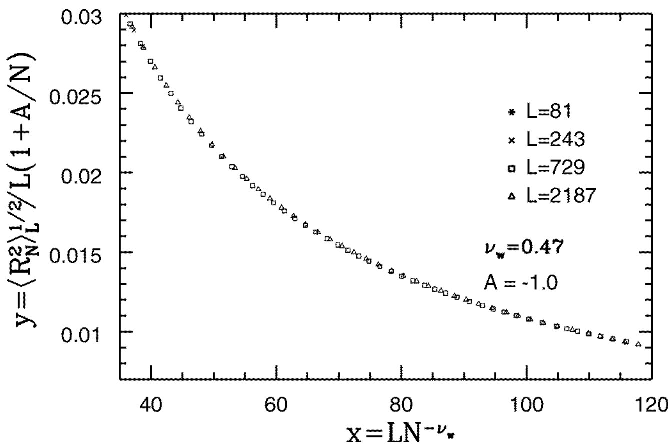

Figure 3. Plot of $y = \langle R_N^2\rangle_L^{1/2}/L(1 + A/N)$ against $x = LN^{-\nu_w}$ for random walks on finite stages of a fractal with generator 1, in the range $35 \leqslant x \leqslant 120$.

When equation (3) is considered, the collapse of these data occurs only in limited ranges of the variable x, because N must simultaneously be large, so that $A/N \approx 0$, and satisfy relation (7). It is possible only using very large lattices.

We note that, if only one member of this ensemble of regular random fractals is considered (corresponding to a particular distribution of lacunas), the data for different values of L collapse into a single curve using the same ν_w. It indicates that the use of large lattices also ensures that the walkers will visit a large number of different microscopic environments, thus representing approximately the whole ensemble of fractals.

For the ensemble of fractals with generator 2, 20 members were considered for the averaging process. In figures 4 and 5 we plot y against x for these lattices, using $\nu_w = 0.445$ and $A = -1.5$.

In table 1 we show the final estimates of ν_w and the corresponding estimates of D_w for the two ensembles of regular random fractals. The error bars are obtained from plots for

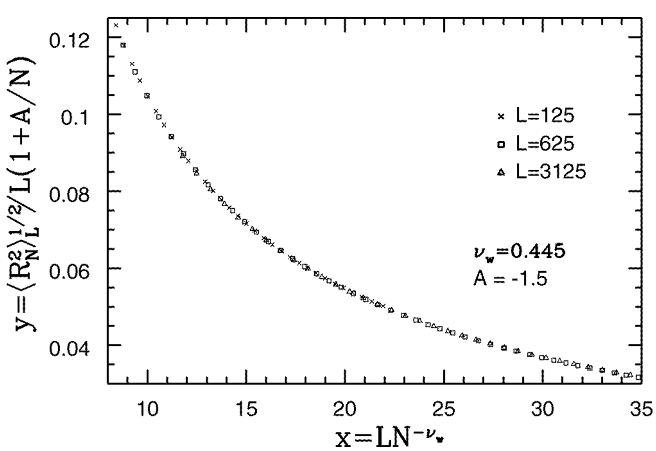

Figure 4. Plot of $y = \langle R_N^2\rangle_L^{1/2}/L(1 + A/N)$ against $x = LN^{-\nu_w}$ for random walks on finite stages of a fractal with generator 2, in the range $8 \leqslant x \leqslant 35$.

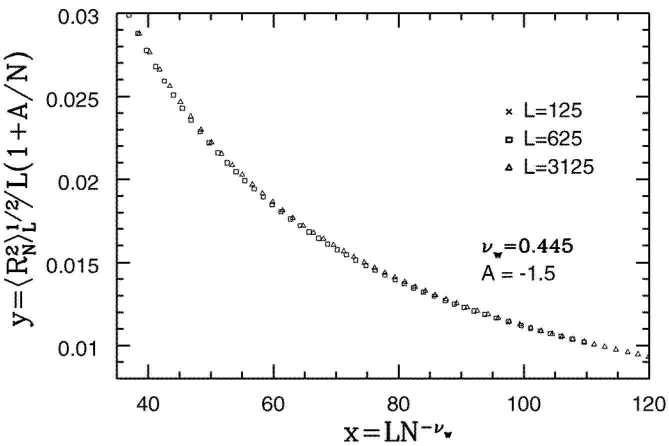

Figure 5. Plot of $y = \langle R_N^2\rangle_L^{1/2}/L(1 + A/N)$ against $x = LN^{-\nu_w}$ for random walks on finite stages of a fractal with generator 2, in the range $35 \leqslant x \leqslant 120$.

different values of ν_w and the constant A, and considering the errors in $\langle R_N^2\rangle_L$. The results for the corresponding Sierpinski carpets are also shown in table 1.

3. Discussion and conclusions

The estimate of ν_w for the regular random fractals are slightly smaller than the estimates for the corresponding carpets (see table 1). This weak dependence of ν_w on randomness motivates the use of deterministic fractals (or the corresponding regular random fractals) as approximate models for real fractals. It is also important for the study of the dependence of ν_w on the fractal geometry.

As previously shown with calculations on Sierpinski carpets, lacunarity has a very small effect on ν_w [4]. Lacunarity measures the inhomogeneity of the distribution of mass in the fractal [12], and the critical behaviour of other statistical systems, like self-avoiding walks, depend on it [13]. Although geometric properties other than D_F and lacunarity certainly influence ν_w, no relation valid for many classes of fractals was proposed. In order to address this question, we present in table 2 the estimates of ν_w for the random regular fractals studied here and for some other finitely ramified structures (statistical and deterministic), covering a large range of values of D_F.

We note that the presence of dead ends (or dangling ends) is crucial for the delay of the diffusion. They are parts of the structures where the walk is forced to return over its own path. Among those fractals, dead ends occur in two-dimensional percolation clusters [14, 15], DLA [16, 17] and the T-fractal [4]. Each of these fractals may be compared to the other ones with similar D_F (see table 2), which do not have this property and clearly have greater ν_w. The presence of lacunas that restrict the path of the walk is responsible for the anomalous diffusion on the structures without dead ends, but their effect is not as strong.

In random regular fractals with generator 2 and the generalized Sierpinski gasket with scale factor $b = 8$ [18], with very similar D_F, the difference of ν_w is relatively small ($\approx$ 3%). This difference may be explained by the ramification, respectively infinite and finite. On the infinitely ramified fractals, the walks have many paths to contour the lacunas, in opposition to the finitely ramified ones, so the diffusion is slightly easier in the former.

Although it seems to be impossible to obtain a complete definition of universality classes

Table 2. Fractal dimensions D_F and anomalous diffusion exponent ν_w for several fractals, including the regular random fractals with generators 1 and 2. Numbers (1), (2) and (3) indicate sets of fractals with similar or equal D_F.

Fractal	D_F	ν_w
(1) Regular random fractals 1	1.892 ...	0.470 ± 0.005
(1) 2-D percolation clusters	1.895 ... [a]	0.3483 ± 0.0001 [b]
(2) Regular random fractals 2	1.722 ...	0.445 ± 0.005
(2) Sierpinski gasket with $b = 8$	1.723 [c]	0.4304 ... [c]
(2) 2-D DLA	1.712 ± 0.003 [d]	0.379 ± 0.007 [e]
(3) Sierpinski gasket with $b = 2$	1.584 ... [f]	0.4306 ... [f]
(3) T-fractal	1.584 ... [g]	0.3868 ... [g]

[a] [14]
[b] [15]
[c] ν_w is obtained from the estimate of D_s [18] and the scaling relation $D_s = 2D_F\nu_w$ [11].
[d] [16]
[e] [17]
[f] [2]
[g] [3]

for random walks or other statistical systems on fractals, the above discussion provides a basis to relate geometric properties to ν_w (or D_w), with a reasonable precision, in structures with similar D_F. It may guide investigations on models for real self-similar structures. For example, it was proposed that silica aerogels should be represented by infinitely ramified fractals, the Sierpinski pastry shells [7]. Regular random fractals constructed from generators of deterministic fractals may be used to model such structures, with some advantages over other statistical fractals, such as the easier rules of construction and exactly known D_F.

We suppose this work will bring new perspectives to the study of critical phenomena on fractal substrates and related fields, by showing that regular random fractals are useful tools to generalize results obtained in deterministic fractals and presenting relations between the fractal geometry and the anomalous diffusion exponent ν_w.

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